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The hydrodynamics of M-theory

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Abstract: We consider the low energy limit of a stack of N M-branes at finite temperature. In this limit, the M-branes are well described, via the AdS/CFT correspondence, in terms of classical solutions to the eleven dimensional supergravity equations of motion. We calculate Minkowski space two-point functions on these M-branes in the long-distance, low-frequency limit, i.e. the hydrodynamic limit, using the prescription of Son and Starinets hep-th/0205051. From these Green's functions for the R-currents and for components of the stress-energy tensor, we extract two kinds of diffusion constant and a viscosity. The N dependence of these physical quantities may help lead to a better understanding of M-branes.

Keywords: M-Theory, AdS-CFT and dS-CFT Correspondence.

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Contents

1. Introduction

The interacting, superconformal field theories (SCFT) living on a stack of N M2- or M5-branes are not well understood. An improved understanding of these M-branes should lead eventually to a better understanding of M-theory itself, a theory that encompasses all the different super string theories and is one of the best hopes for a quantum theory of gravity. While the full M-brane theories remain mysterious, the low energy, large N behavior is conjectured to be described well, via the AdS/CFT correspondence [1]–[3], by certain classical solutions to eleven dimensional supergravity equations of motion. Recent work on AdS/CFT correspondence by Son, Starinets, and Policastro [4, 5] provides a prescription for writing Minkowski space two-point functions for these types of theories. We take advantage of this prescription to calculate viscosities and diffusion constants for M-brane theories in this low energy limit, thus generalizing the work of [5] for D3-branes.

Policastro, Son, and Starinets [5] used their Minkowski space prescription to investigate the low-frequency, long-distance, finite temperature regime of the D3-brane theory. There is lore that the long-distance, low-frequency behavior of any interacting theory at finite temperature can be described well by fluid mechanics (hydrodynamics) [6]. Although not rigorously proven, the idea is well supported by physical intuition about macroscopic systems. Hydrodynamics in turn provides rigorous constraints for the form of Minkowski space correlation functions. Once a few viscosities, diffusion constants, and other transport coefficients are known, the two-point functions are completely fixed [7].

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Indeed, the authors of [5] found that the form of the Green's functions calculated from supergravity was completely consistent with hydrodynamics for this D3-brane theory. Moreover, from these Green's functions, they were able to extract a transport coefficient (a shear viscosity) and diffusion constants. The authors' prescription for Minkowski space Green's functions is a modification of the prescription for calculating euclidean space Green's function developed in [2, 3]. Both prescriptions use the gravitational description of the low energy theory on the brane for calculating correlation functions.

It is true that these same Green's functions on a D3-brane can be calculated in a more traditional way using weakly coupled gauge theory [8]. The low energy theory living on a stack of N D3-branes is described via the AdS/CFT correspondence, alternately as $\mathcal{N} = 4$ SU(N) super Yang Mills or as type IIB supergravity in a $AdS_5 \times S^5$ background. While the gravity calculation is good at strong 't Hooft coupling λ (large curvature), the gauge theory calculation is good only for small λ. However, the N and temperature dependence of these transport coefficients and diffusion constants should be and indeed is universal and independent of the method of calculation. Remarkably, the gravity calculation appears to be technically simpler than the equivalent calculation for weakly coupled gauge theory.

In contrast, the M-brane theories have no such alternate gauge theory description. However, there are still low-energy supergravity solutions from which we can extract analogous Minkowski space Green's functions. Moreover, we can hope that M-branes, like D3-branes, have a hydrodynamic regime.

Indeed, based on the Minkowski space prescription of Son and Starinets [4], in the low-frequency, long-distance, finite temperature regime, we find that these M-brane theories have two-point functions which are completely consistent with a hydrodynamic interpretation. Generalizing [5], we calculate two-point functions for the conserved R-symmetry current and for components of the stress-energy tensor for these M-branes. Moreover, from these Green's functions we are able to extract corresponding diffusion constants and also the viscosity. The N dependence of these physical quantities may lead to a better understanding of M-brane theories.

It should be emphasized that the Son and Starinets prescription [4] for calculating Minkowski space correlators is not completely justified, and that this paper provides some limited additional evidence for their prescription. Turning the logic of the last two paragraphs around, we can argue that on general grounds we expected the M-branes to have a hydrodynamic description. Thus, it is reassuring that the prescription of [4] does indeed produce Green's functions with the appropriate behavior. Also, we get results which are internally consistent. On general hydrodynamic grounds, we expect that $D = \eta/(\epsilon + P)$ where D is the diffusion constant calculated from the stress energy tensor, η the viscosity, ϵ the energy density, and P the pressure. We check that this equation holds both for the M2-branes and M5-branes.

We begin by reviewing some essential facts about the non-extremal M2- and M5-brane backgrounds. As we are working at finite temperature, the extremal $AdS_4 \times S^7$ and $AdS_7 \times S^4$ supergravity solutions are not adequate. We need their nonextremal generalizations where we can associate a Hawking temperature to the horizon.

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Next, we consider R-current correlation functions. Both M2- and M5-brane supergravity solutions have an R-symmetry which one can think of roughly as the rotational symmetry of the transverse sphere. From the form of the thermal R-current two-point functions, we extract a corresponding R-charge diffusion constant.

Having warmed up with the R-current correlators, we proceed to the more complicated example of stress-energy tensor two-point functions. From components of these two-point functions, we extract a diffusion constant and a viscosity for each M-brane theory.

Finally, we end with some comments about the N dependence or lack thereof of the various diffusion constants and transport coefficients calculated. The motivation for this work came in large part out of the hope that some of these N dependences might shed light on the underlying M-brane theories.

2. The nonextremal eleven dimensional supergravity backgrounds

We begin by reviewing some essential facts about the non-extremal M2- and M5-brane supergravity solutions. Roughly speaking, these solutions represent what happens when a stack of M-branes is placed in flat 11-dimensional space and given some finite temperature T. The 11-dimensional space, close to the M-branes, separates into a product of a sphere and an asymptotically anti-de Sitter space. A horizon with Hawking temperature T forms. These supergravity solutions are solutions to the equations of motion following from the eleven dimensional supergravity action [9]

$$\frac{1}{2\kappa_{11}^2}\int d^{11}x(-g)^{1/2}\mathcal{R} - \frac{1}{4\kappa_{11}^2}\int\left(F_4\wedge\star F_4 + \frac{1}{3}A_3\wedge F_4\wedge F_4\right), \tag{2.1}$$

where κ_{11} is the gravitational coupling strength, and $dA_3 = F_4$.

2.1 M5-brane

For the M5-brane, the nonextremal metric is

$$ds^2 = H(r)^{-1/3}\left[-f(r)dt^2 + d\vec{x}^2\right] + H(r)^{2/3}\left[\frac{dr^2}{f(r)} + r^2 d\Omega_4^2\right], \tag{2.2}$$

where $H(r) = 1 + R^3/r^3$ and $f(r) = 1 - r_0^3/r^3$. The quantity $d\vec{x}^2$ is a metric on flat, euclidean $\mathbb{R}^5$. The term $d\Omega_4^2$ is the metric on a unit four sphere S^4. The four form flux F_4 from the M5-branes threads this S^4:

$$F_4 = 3R^3\mathrm{vol}(S^4)\,. \tag{2.3}$$

The quantization condition on the flux implies that $N^3\kappa_{11}^2 = 2^7\pi^5R^9$ where N is the number of M5-branes and κ_{11} is the eleven dimensional gravitational coupling strength [10].

Taking the near-horizon limit $r \ll R$, according to the AdS/CFT prescription [1, 2, 3] we can "zoom in" on the five-brane theory dynamics:

$$ds^2 \to \frac{r_0}{uR}\left[-f(u)dt^2 + d\vec{x}^2\right] + \frac{R^2}{f(u)}\frac{du^2}{u^2} + R^2 d\Omega_4^2\,. \tag{2.4}$$

We have made the coordinate transformation $u = r_0/r$. Spatial infinity, which is also now the boundary of an asymptotically anti-de Sitter space, corresponds to $u = 0$. There is a horizon at $u = 1$. The Hawking temperature of this horizon is

$$T = \frac{3}{4\pi} \frac{r_0^{1/2}}{R^{3/2}} . \tag{2.5}$$

Another important quantity characterizing this supergravity solution is the entropy density, which one finds by multiplying the horizon area by $2\pi/\kappa_{11}^2$ and dividing out by the volume of the gauge theory directions, $x^1, x^2, \ldots, x^5$ [11]:

$$S = \frac{2^7 \pi^3}{3^6} N^3 T^5 . \tag{2.6}$$

We will use index conventions where $\mu, \nu, \ldots$ refer to the asymptotically AdS directions, $\alpha, \beta, \ldots$ index the M-brane directions only, and $i, j, \ldots$ index the spatial M-brane directions.

2.2 M2-brane

An analogous nonextremal supergravity solution exists for a stack of M2-branes in eleven dimensional space. Now the metric takes the form

$$ds^2 = H(r)^{-2/3} \left[-f(r) dt^2 + dx^2 + dy^2 \right] + H(r)^{1/3} \left[\frac{dr^2}{f(r)} + r^2 d\Omega_7^2 \right] , \tag{2.7}$$

where $H(r) = 1 + R^6/r^6$ and $f(r) = 1 - r_0^6/r^6$. The four form field strength is easier to think of in a dual language as a seven form field strength:

$$\star F_4 = F_7 = 6R^6 \mathrm{vol}(S^7) . \tag{2.8}$$

The quantization condition [10] on the field strength reveals that $R^9 \pi^5 = N^{3/2} \kappa_{11}^2 \sqrt{2}$.

We can again take a near horizon limit, $r \ll R$, to find

$$ds^2 \to \frac{r_0^4}{u^2 R^4} \left[-f(u) dt^2 + dx^2 + dy^2 \right] + \frac{R^2}{4f(u)} \frac{du^2}{u^2} + R^2 d\Omega_7^2 , \tag{2.9}$$

where $u = r_0^2/r^2$. The Hawking temperature is

$$T = \frac{3}{2\pi} \frac{r_0^2}{R^3} , \tag{2.10}$$

and the entropy density is [11]

$$S = \frac{8\sqrt{2}\pi^2}{27} N^{3/2} T^2 . \tag{2.11}$$

3. R-charge diffusion for M2- and M5-branes

The R-charge interactions are mediated in the bulk by a gauge field $F^a_{\mu\nu}$. The starting point for calculating two-point R-charge correlators is the usual Maxwell action in the nonextremal backgrounds given above:

$$S = -\frac{1}{4g_{SG}^2} \int d^d x \sqrt{-g} F^a_{\mu\nu} F^{\mu\nu\, a} \,. \tag{3.1}$$

Here d is equal to 4 for the M2-branes and 7 for the M5-branes. The tensor $g_{\mu\nu}$ is a metric on the asymptotically anti-de Sitter space. The calculation is similar to the euclidean calculations in [12]; however we work with a lorentzian signature and in a nonextremal background. The constant g_{SG} can be set by compactifying the eleven dimensional supergravity action (2.1) on a S^4 or S^7. We will ignore this overall normalization for the Green's functions for now. The reason is that our main interest in this section is the diffusion coefficient for the R-current which can be obtained simply from the location of the pole in the corresponding retarded Green's functions. The overall normalization of the Green's function is also related to the diffusion constant, but in a more complicated way via a Kubo-type formula.

Our calculations closely follow [5]. We work in the gauge $A_u = 0$. We use a Fourier decomposition

$$A_\mu = \int \frac{d^{d-1}q}{(2\pi)^{d-1}} e^{-i\omega t + i\mathbf{q}\cdot\mathbf{x}} A_\mu(q,u) \,. \tag{3.2}$$

Rotational invariance in the spatial directions allows one to simplify things further by choosing $q^0 = \omega$, $q^1 = q$, and all other $q^i = 0$. The equations of motion for the A_μ are

$$\frac{1}{\sqrt{-g}} \partial_\nu \left[\sqrt{-g} g^{\mu\rho} g^{\nu\sigma} \left(\partial_\rho A_\sigma - \partial_\sigma A_\rho \right) \right] = 0 \,. \tag{3.3}$$

At this point, it becomes convenient to analyze the M2- and M5-brane cases separately.

3.1 R-charge and M5-branes

The equations of motion (3.3) for A_μ reduce to

$$\omega_5 A_t' + f q_5 A_x' = 0 \,, \tag{3.4}$$

$$A_t'' - \frac{1}{u} A_t' - \frac{1}{uf} \left(\omega_5 q_5 A_x + q_5^2 A_t \right) = 0 \,, \tag{3.5}$$

$$A_x'' - \frac{1}{u} A_x' + \frac{f'}{f} A_x' + \frac{1}{uf^2} \left(q_5 \omega_5 A_t + \omega_5^2 A_x \right) = 0 \,, \tag{3.6}$$

$$A_\alpha'' - \frac{1}{u} A_\alpha' + \frac{f'}{f} A_\alpha' + \frac{1}{uf^2} \left(\omega_5^2 - f q_5^2 \right) A_\alpha = 0 \,, \tag{3.7}$$

where t is x^0, x is x^1, and α stands for any of the other x^i. We have introduced

$$\omega_5 \equiv \frac{3}{4\pi T} \omega \,; \qquad q_5 \equiv \frac{3}{4\pi T} q \,. \tag{3.8}$$

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The prime denotes the derivative with respect to u. The peculiar combinations of q_5^2, $\omega_5 q_5$, and q_5^2 in the equations guarantee gauge invariance under the residual transformation $A_t \to A_t - \omega\Lambda$ and $A_x \to A_x + q\Lambda$. The first three equations are dependent; equations (3.4) and (3.5) imply equation (3.6).

Because of our choice of q-vector, the R-charge diffusion appears only in the A_t and A_x sector, and we begin with the first two equations, (3.4) and (3.5). These two equations can be combined to yield a single equation for A_t':

$$A_t''' + \frac{f'}{f} A_t'' + \frac{\omega_5^2 - q_5^2 f - f f'}{u f^2} A_t' = 0 \,. \tag{3.9}$$

This second order equation for A_t' does not appear to be analytically tractable. However, a solution can be obtained perturbatively for small q_5 and ω_5. Following [5], we determine the behavior of A_t' near the singular point $u = 1$. Substituting $A_t' = (1-u)^\alpha F(u)$, where $F(u)$ is a regular function, one finds that $\alpha^2 = -\omega_5^2/9$. The "incoming wave" boundary condition described in [4] forces us to choose $\alpha = -i\omega_5/3$.

Next, we solve for $F(u)$ perturbatively:

$$F(u) = F_0(u) + \omega_5 F_1(u) + q_5^2 G_1(u) + \mathcal{O}(\omega_5^2, \omega_5 q_5^2, q_5^4) \,, \tag{3.10}$$

where $F_0(u) = uC$ and

$$\begin{aligned} F_1(u) &= iC \left[(u-1) + \frac{u}{6} f_1(u) - \frac{u}{\sqrt{3}} f_2(u) \right] , \\ G_1(u) &= C \left[\frac{1}{2}(1-u) + \frac{u}{\sqrt{3}} f_2(u) \right] , \end{aligned} \tag{3.11}$$

where

$$f_1(u) = \ln \frac{1+u+u^2}{3} \,; \qquad f_2(u) = \tan^{-1}\left(\frac{1+2u}{\sqrt{3}}\right) - \frac{\pi}{3} \,. \tag{3.12}$$

One of the two integration constants for $F_0(u)$ is set by requiring that $F_0(u)$ is well-behaved at $u = 1$. The integration constants for all higher $F_i(u)$ and $G_i(u)$ are set by requiring that these functions vanish at $u = 1$ as well.

The constant C can be related to the boundary values A_t^0 and A_x^0 using (3.5):

$$C = \frac{\omega_5 q_5 A_x^0 + q_5^2 A_t^0}{i\omega_5 - (1/2) q_5^2} \,. \tag{3.13}$$

The pole in C is the same pole that appears in the retarded Green's functions, as we will presently see. Having obtained A_t', A_x' follows from (3.4).

The solution to (3.7) can be obtained in a similar fashion:

$$A_\alpha = \frac{A_\alpha^0 (1-u)^{-i\omega_5/3} h(u)}{h(0)} + \mathcal{O}(\omega_5^2, \omega_5 q_5^2, q_5^4) \,, \tag{3.14}$$

where

$$h(u) = 1 + i\omega_5 \left(\frac{1}{6} f_1(u) - \frac{1}{\sqrt{3}} f_2(u) \right) - q_5^2 \frac{2}{\sqrt{3}} f_2(u) \,. \tag{3.15}$$

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The Green's functions can now be calculated from the terms in the action which contain two derivatives with respect to u:

$$\begin{aligned} S &= -\frac{1}{2g_{SG}^2}\int du\, d^6x\, \sqrt{-g}g^{uu}g^{ij}\partial_u A_i \partial_u A_j + \cdots \\ &= \frac{r_0^2}{2R^3 g_{SG}^2}\int du\, d^6x\, \frac{1}{u}\left[A_t'^2 - f\sum_{i=1}^{5} A_{x^i}'^2\right]. \end{aligned} \tag{3.16}$$

Recall from [4] the procedure for calculating these Minkowski space Green's functions for a scalar $\phi(u)$. We extract the function $A(u)$ that multiplies $(\partial_u\phi)^2$ in the action:

$$S = \frac{1}{2}\int du\, d^6x\, A(u)(\partial_u\phi)^2 . \tag{3.17}$$

Next, we express the bulk field ϕ via its value ϕ_0 at the boundary $u = 0$, $\phi(u,q) = f_q(u)\phi_0(q)$. By definition $f_q(0) = 1$. Moreover, we impose an incoming-wave boundary condition on $f_q(u)$ at the horizon $u = 1$ (when q is timelike). The retarded (Minkowski space) Green's function is then defined to be

$$G^R(q) = A(u)f_{-q}(u)\partial_u f_q(u)\big|_{u=0} . \tag{3.18}$$

Using this prescription, we find that the retarded Green's functions for the R-current are

$$G^{ab}_{tt} = \mathcal{C}\frac{q^2\delta^{ab}}{i\omega - D_R q^2} + \cdots , \tag{3.19}$$

$$G^{ab}_{xt} = G^{ab}_{tx} = -\mathcal{C}\frac{\omega q\delta^{ab}}{i\omega - D_R q^2} + \cdots , \tag{3.20}$$

$$G^{ab}_{xx} = \mathcal{C}\frac{\omega^2\delta^{ab}}{i\omega - D_R q^2} + \cdots , \tag{3.21}$$

$$G^{ab}_{\alpha\alpha} = -\mathcal{C}\delta^{ab}(i\omega + 2D_R q^2) + \cdots , \tag{3.22}$$

where

$$\mathcal{C} = \frac{r_0^{3/2}}{R^{3/2}g_{SG}^2} \sim N^3T^3 , \tag{3.23}$$

and

$$D_R = \frac{3}{8\pi T} . \tag{3.24}$$

As expected, the $G_{\alpha\alpha}$ Green's functions have no pole while the others do. From the location of the pole, we can read off the diffusion coefficient for the R-charge, D_R. We regard this value of D_R as a prediction for the theory living on a stack of M5-branes at finite temperature. The power of T in D_R is forced by dimensional analysis. However, it is interesting that this value for D_R is N independent.

This expression for D_R is subject to two kinds of correction. First, the supergravity approximates these M-branes well only at large N. Thus, there could be $1/N$ corrections. The second correction is not really a correction to D_R but to the location of the pole itself.

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We could easily calculate the A_μ to higher order in q_5 and ω_5. In this case, we would get corrections of order $\mathcal{O}(q_5^2)$ to the location of the pole.

Although we did not solve for g_{SG} exactly, we can count powers of N and T in $\mathcal{C}$. The power of T is forced by dimensional analysis. Tracing powers of N is relatively easy. The coupling g_{SG} is essentially κ_{11} multiplied by lots of N independent compactification factors, and $\kappa_{11} \sim N^{-3/2}$.

3.2 R-charge and M2-branes

We now redo this same calculation in the nonextremal M2-brane background (2.9). The differential equations for A_μ take the modified form

$$\omega_2 A_t' + q_2 f A_x' = 0\,, \tag{3.25}$$

$$A_t'' - \frac{1}{4f}\left(\omega_2 q_2 A_x + q_2^2 A_t\right) = 0\,, \tag{3.26}$$

$$A_x'' + \frac{f'}{f}A_x' + \frac{1}{4f^2}\left(\omega_2 q_2 A_t + \omega_2^2 A_x\right) = 0\,, \tag{3.27}$$

$$A_y'' + \frac{f'}{f}A_y' + \frac{1}{4f^2}\left(\omega_2^2 - f q_2^2\right) A_y = 0\,, \tag{3.28}$$

where $x^0 = t$, $x^1 = x$, and $x^2 = y$. We have defined the quantities

$$\omega_2 \equiv \frac{3}{2\pi T}\omega\,; \qquad q_2 \equiv \frac{3}{2\pi T}q\,. \tag{3.29}$$

This system is very similar to the one encountered in the previous section. Equations (3.25) and (3.26) imply equation (3.27). We combine equations (3.25) and (3.26) to give a single differential equation for A_t' alone:

$$A_t''' + \frac{f'}{f}A_t'' + \frac{1}{4f^2}(\omega_2^2 - f q_2^2)A_t' = 0\,. \tag{3.30}$$

We make the substitution $A_t' = (1-u)^{-i\omega_2/6}F(u)$ and solve for $F(u)$ perturbatively in ω_2 and q_2^2:

$$F(u) = C(1 + \omega_2 F_1(u) + q_2^2 G_1(u) + \cdots)\,, \tag{3.31}$$

where

$$\begin{aligned} F_1(u) &= \frac{i}{12}f_1(u) + \frac{i}{2\sqrt{3}}f_2(u)\,, \\ G_1(u) &= -\frac{1}{2\sqrt{3}}f_2(u)\,. \end{aligned} \tag{3.32}$$

Using (3.26), we can solve for C in terms of the boundary values of A_t and A_x:

$$C = \frac{\omega_2 q_2 A_x^0 + q_2^2 A_t^0}{2(i\omega_2 - (1/2)q_2^2)}\,. \tag{3.33}$$

The pole in C will be the same pole that appears in the R-current Green's functions and hence is related to the diffusion coefficient.

The perturbative expression for A'_x can be obtained from (3.25). We have already done the necessary work for calculating A_y. Note that (3.28) is the same differential equation as (3.30). Thus $A_y = (1-u)^{-i\omega_2/6}F(u)$ with $F(u)$ given by (3.31). The only change is that now $C = A_y^0 + \mathcal{O}(\omega_2, q_2^2)$.

To extract the Green's functions, we need to isolate the terms in the action with two u derivatives:

$$S = \frac{r_0^2}{R^3 g_{SG}^2} \int du\, d^3x \left[A_t'^2 - fA_x'^2 - fA_y'^2 + \cdots\right] . \tag{3.34}$$

From this expression, it is straightforward to see that

$$\begin{aligned}
G_{tt}^{ab} &= \frac{q^2 \delta^{ab}}{g_{SG}^2(i\omega - D_R q^2)} , \\
G_{xt}^{ab} &= G_{tx}^{ab} = -\frac{\omega q \delta^{ab}}{g_{SG}^2(i\omega - D_R q^2)} , \\
G_{xx} &= \frac{\omega^2 \delta^{ab}}{g_{SG}^2(i\omega - D_R q^2)} , \\
G_{yy}^{ab} &= -\frac{\delta^{ab}}{g_{SG}^2}\left(i\omega - D_R q^2\right) ,
\end{aligned} \tag{3.35}$$

where $1/g_{SG}^2 \sim N^{3/2}$ and the diffusion coefficient is

$$D_R = \frac{3}{4\pi T} . \tag{3.36}$$

This expression for D_R is again N independent.

4. Stress energy two point functions

To obtain more diffusion constants for our M-brane theory in the hydrodynamic limit, we compute the two-point function of the stress-energy tensor. According to the AdS/CFT prescription, this two-point function is related to small perturbations of the metric in the bulk theory. In particular, we consider $g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu}$ where the unperturbed metric $g_{\mu\nu}^{(0)}$ is given by the asymptotically AdS pieces of the full eleven dimensional metrics (2.9) and (2.4). The transverse spherical parts of the eleven dimensional metrics we leave unchanged.

For this calculation, we can ignore the spherical piece of the metric and the four form field strength F_4. They do not enter at first order in the perturbations. For the asymptotically AdS_7 case, i.e. the five-brane case, we consider the compactified action

$$S = \frac{R^4 \mathrm{Vol}(S^4)}{2\kappa_{11}^2}\left[\int du\, d^6x \sqrt{-g}(\mathcal{R} - 2\Lambda) + 2\int d^6x \sqrt{-g^B} K\right] . \tag{4.1}$$

In the above, $\mathcal{R}$ is the Ricci scalar and $\Lambda = -15/4R^2$ is a cosmological constant arising from the integral over $|F_4|^2$ in the full eleven dimensional action. The metric $g_{\alpha\beta}^B$ is the metric induced on the boundary $u = 0$ while K is the extrinsic curvature of the boundary. There is an analogous action for the asymptotically AdS_4 case.

$$S = \frac{R^7 \mathrm{Vol}(S^7)}{2\kappa_{11}^2}\left[\int du\, d^3x \sqrt{-g}(\mathcal{R} - 2\Lambda) + 2\int d^3x \sqrt{-g^B} K\right] , \tag{4.2}$$

where the cosmological constant is now $\Lambda = -12/R^2$ instead.

The first step in computing the stress-energy two-point function is to solve the linearized Einstein's equations

$$\mathcal{R}^{(1)}_{\mu\nu} = \frac{2}{d-2}\Lambda h_{\mu\nu} , \tag{4.3}$$

where d is either 4 (for asymptotically AdS_4) or 7 (for asymptotically AdS_7). By $\mathcal{R}^{(1)}_{\mu\nu}$, we mean everything in the Ricci curvature $\mathcal{R}_{\mu\nu}$ that is linear in $h_{\mu\nu}$.

The metric perturbations split up naturally into groups as follows. We take a Fourier decomposition of the metric perturbation, assuming that $h_{\mu\nu}$ depends on t and x as $e^{-i\omega t+iqx}$, defining $x^0 = t$, $x^1 = x$, $x^2 = y$, and so on. We choose a gauge such that $A_{\mu u} = 0$. For the M5-brane case, there is a rotation group SO(4) acting on the directions transverse to u, t, and x. The perturbations split according to whether $h_{\mu\nu}$ is a tensor, a vector, or a scalar under these rotations. For the asymptotically AdS_4 case, there is only one direction, the y direction, transverse to the others. However, we can still consider the effect of sending $y \to -y$. Under this transformation, the metric perturbations split into two groups according to whether $h_{\mu\nu}$ has an odd or an even number of y indices. Stretching definitions, we will call the metric perturbation with only one y index a vector mode. There is one remaining perturbation involving components of $h_{\mu\nu}$ with both no and two y indices which we will call the scalar mode.

The modes are useful in different ways. The tensor mode allows us to compute a shear viscosity of the boundary theory using a Kubo formula. There is a diffusion pole in the vector mode which will allow us to calculate a diffusion constant. There is also a Kubo formula for these vector modes which will allow for another check of the viscosity. The scalar mode describes sound propagation on the boundary.

We will begin by computing the diffusion constant from the poles in the vector modes, both for asymptotically AdS_4 and AdS_7. The calculation is very similar to that of the R-charge diffusion constant. The relation should not be at all surprising because from an eleven dimensional standpoint, the A_μ vector potential is is in part a metric perturbation of the form $h_{M\mu}$ where M is a spherical direction and μ is an asymptotically AdS direction. Next, we will analyze the tensor mode for asymptotically AdS_7. We leave sound propagation for future work.

Sound propagation is the most involved of the three types of metric fluctuations. To consider a tensor fluctuation for the M5-brane, h_{yz} is the only component of the metric fluctuations that needs to be nonzero. For the vector fluctuations, two components of $h_{\mu\nu}$ need to be nonzero, for example h_{xy} and h_{ty}. For the scalar modes, all of the diagonal components of $h_{\alpha\beta}$ plus h_{xt} must be nonzero. The resulting system of differential equations is less tractable than for the tensor or vector modes.

4.1 Vector modes and a diffusion constant for M5-branes

We consider a metric perturbation of the form $h_{ty} \neq 0$ and $h_{xy} \neq 0$ with all the other $h_{\mu\nu} = 0$. Moreover, we make a Fourier decomposition such that

$$\begin{aligned} h^y_x &= e^{-i\omega t+iqx} H_x(u) , \\ h^y_t &= e^{-i\omega t+iqx} H_t(u) . \end{aligned} \tag{4.4}$$

The linearized Einstein's equations for H_x and H_t are

$$\omega_5 H_t' + q_5 H_x' f = 0\,, \tag{4.5}$$

$$H_t'' - \frac{2}{u}H_t' - \frac{1}{uf}(\omega_5 q_5 H_x + q_5^2 H_t) = 0\,, \tag{4.6}$$

$$H_x'' - \frac{3-f}{uf}H_x' + \frac{1}{uf^2}(\omega_5 q_5 H_t + \omega_5^2 H_x) = 0\,. \tag{4.7}$$

This system of equations is very similar to the systems (3.4)–(3.6) and (3.25)–(3.27) we solved in the previous sections and is tractable using exactly the same methods. We combine the first two equations (4.5) and (4.6) to get a single equation for H_t':

$$H_t''' + \frac{2f-3}{uf}H_t'' + \frac{1}{uf^2}(\omega_5^2 - fq_5^2 + 6u^2 f)H_t' = 0\,. \tag{4.8}$$

We make the ansatz $H_t' = C(1-u)^{-i\omega_5/3}F(u)$ and solve for the regular function $F(u)$ perturbatively in ω_5 and q_5:

$$F(u) = u^2 + i\omega_5\left(\frac{1}{2}(u^2-1) + \frac{1}{6}u^2 f_1(u) + \frac{u^2}{\sqrt{3}}f_2(u)\right) + \frac{q_5^2}{6}(1-u^2)\,. \tag{4.9}$$

Taking the limit $u \to 0$, we solve for C in terms of H_x^0 and H_t^0 using (4.6):

$$C = \frac{\omega_5 q_5 H_x^0 + q_5^2 H_t^0}{i\omega_5 - (1/3)q_5^2}\,. \tag{4.10}$$

To get the two-point functions, we need to isolate the terms in the action proportional to $H_x'^2$ and $H_t'^2$:

$$S = \frac{2^6\pi^3}{3^7}N^3T^6\int du\, d^6x\,\frac{1}{u^2}\left[H_t'^2 - fH_x'^2 + \cdots\right]. \tag{4.11}$$

A subtle point can be made about the Gibbons-Hawking term here. In order to isolate these $H_\alpha'^2$ terms in the action, we have integrated by parts terms of the form $H_\alpha H_\alpha''$. During this integration by parts, boundary terms of the form $H_\alpha H_\alpha'$ appear at $u=0$. One might think that these boundary terms will affect the overall normalization of the two-point functions, but they can't. The reason is that the Gibbons-Hawking term is precisely of a form that cancels these particular boundary contributions. Recall that the extrinsic curvature is defined to be

$$K = -\nabla^\mu n_\mu\,, \tag{4.12}$$

where n_μ is a unit vector, normal to the boundary $u=0$. There is a piece of K that looks like

$$2\sqrt{-g^B}K = -\sqrt{-g}g^{uu}g^{\alpha\beta}g_{\alpha\beta,u}\Big|_{u=0} + \cdots\,, \tag{4.13}$$

which precisely cancels the boundary contribution from terms of the form $H_\alpha H_\alpha''$.

From this quadratic piece of the action, we read off the retarded Green's functions:

$$\begin{aligned} G_{ty,ty} &= \frac{2^5\pi^2N^3T^5q^2}{3^6(i\omega - Dq^2)}\,, \\ G_{ty,xy} &= -\frac{2^5\pi^2N^3T^5\omega q}{3^6(i\omega - Dq^2)}\,, \\ G_{xy,xy} &= \frac{2^5\pi^2N^3T^5\omega^2}{3^6(i\omega - Dq^2)}\,, \end{aligned} \tag{4.14}$$

where the diffusion coefficient is

$$D = \frac{1}{4\pi T}\,. \tag{4.15}$$

There exists a Kubo formula (see for example [7]) for these two-point functions that will let us calculate the viscosity η. In particular

$$\eta = -\lim_{\omega\to 0}\left[\lim_{q\to 0}\frac{\omega}{q^2}\mathrm{Im}\,G_{ty,ty}\right] = \frac{2^5\pi^2}{3^6}N^3T^5\,. \tag{4.16}$$

There is a relation between the shear viscosity (4.16) and the diffusion constant (4.15): $D = \eta/(\epsilon + P)$, where ϵ is the energy density and P is the pressure. The relation is a consequence of the conservation condition on the stress-energy tensor, $\partial_\alpha T^{\alpha\beta} = 0$, along with a linearized hydrodynamic equation for the purely spatial parts of the stress-energy tensor [6][1]

$$T^{ij} = P\delta^{ij} - \frac{\eta}{\epsilon + P}\left(\partial_i T^{0j} + \partial_j T^{0i} - \frac{2}{3}\delta^{ij}\partial_k T^{0k}\right). \tag{4.17}$$

Putting these two equations together, one finds a diffusion equation for the shear modes, i.e. the fluctuations of the momentum density T^{0i} such that $\partial_i T^{0i} = 0$. The diffusion equation for T^{ty} is $\partial_t T^{ty} = D\partial_x^2 T^{ty}$ where $D = \eta/(\epsilon + P)$.

This relation between η and D will let us check that our calculations are internally consistent. From the thermodynamic relation between the entropy density and the pressure, $S = \partial P/\partial T$, one can calculate P. Because the stress energy tensor is traceless, $\epsilon = 5P$. One finds that indeed $D = \eta/6P = 1/4\pi T$.

4.2 Vector modes and a diffusion constant for M2-branes

The same story can be repeated almost verbatim for the M2-brane case. We take the analogous definition for H_x and H_t, being careful to raise the indices of the h_{ty} and h_{xy} metric perturbations with the appropriate non-extremal M2-brane metric instead. The linearized Einstein's equations for H_x and H_t are

$$\omega_2 H_t' + q_2 H_x' f = 0\,, \tag{4.18}$$

$$H_t'' - \frac{2}{u}H_t' - \frac{1}{4f}(\omega_2 q_2 H_x + q_2^2 H_t) = 0\,, \tag{4.19}$$

$$H_x'' - \frac{3-f}{uf}H_x' + \frac{1}{4f^2}(\omega_2 q_2 H_t + \omega_2^2 H_x) = 0\,. \tag{4.20}$$

Some small changes in the wave-vector dependent pieces of the differential equations are apparent, but otherwise the system is virtually identical to that of (4.5)–(4.7). We combine the first two equations (4.18) and (4.19) to get a single equation for H_t':

$$H_t''' + \frac{f-3}{uf}H_t'' + \frac{1}{4f^2}(\omega_2^2 - fq_2^2)H_t' + \frac{2}{u^2 f}(3-2f)H_t' = 0\,. \tag{4.21}$$

[1] The bulk viscosity in these M-brane models is zero.

We make the ansatz $H_t' = C(1-u)^{-i\omega_2/6}F(u)$ and solve for the regular function $F(u)$ perturbatively in ω_2 and q_2:

$$F(u) = u^2 + i\omega_2\left(\frac{1}{2}(u^2-u) + \frac{1}{12}u^2 f_1(u) - \frac{u^2\sqrt{3}}{6}f_2(u)\right) + \frac{q_2^2}{12}(u-u^2)\,. \tag{4.22}$$

Taking the limit $u \to 0$, we solve for C in terms of H_x^0 and H_t^0 using (4.19):

$$C = \frac{\omega_2 q_2 H_x^0 + q_2^2 H_t^0}{2\left(i\omega_2 - \frac{1}{6}q_2^2\right)}. \tag{4.23}$$

To get the two-point functions, we need to isolate the terms in the action proportional to $H_x'^2$ and $H_t'^2$:

$$S = \frac{2^{5/2}\pi^2}{3^4}N^{3/2}T^3\int du\, d^3x \frac{1}{u^2}\left[H_t'^2 - fH_x'^2 + \cdots\right]. \tag{4.24}$$

From here, we read off the retarded Green's functions:

$$\begin{aligned} G_{ty,ty} &= \frac{2^{3/2}\pi N^{3/2}T^2 q^2}{3^3(i\omega - Dq^2)}\,, \\ G_{ty,xy} &= -\frac{2^{3/2}\pi N^{3/2}T^2 \omega q}{3^3(i\omega - Dq^2)}\,, \\ G_{xy,xy} &= \frac{2^{3/2}\pi N^{3/2}T^2 \omega^2}{3^3(i\omega - Dq^2)}\,, \end{aligned} \tag{4.25}$$

where the diffusion coefficient is again

$$D = \frac{1}{4\pi T}\,. \tag{4.26}$$

Invoking the Kubo formula, one finds from the normalization of the Green's functions that the shear viscosity is

$$\eta = \frac{2^{3/2}\pi}{3^3}N^{3/2}T^2\,. \tag{4.27}$$

We can check that the diffusion constant $D = 1/4\pi T$ is consistent with this formula for the viscosity, $D = \eta/(\epsilon + P)$. Tracelessness of the stress energy tensor now implies that $\epsilon = 2P$. Using the fact that $S = \partial P/\partial T$, one finds that indeed $D = \eta/3P = 1/4\pi T$.

4.3 Tensor perturbations, M5-branes, and a Kubo formula

We consider a perturbation of the non-extremal M5-brane metric of the form $h_{yz} \neq 0$ with all other $h_{\mu\nu} = 0$. We decompose such a perturbation into Fourier modes, defining

$$h_y^z \equiv e^{-i\omega t + iqx}\phi(u)\,. \tag{4.28}$$

The linearized Einstein equation for h_{yz} becomes

$$\phi'' - \frac{2+u^3}{uf}\phi' + \frac{1}{uf^2}(\omega_5^2 - fq_5^2)\phi = 0\,. \tag{4.29}$$

This differential equation can be rewritten more simply as $\Box h_y^z = 0$. Thus this particular metric perturbation can be thought of as a massless scalar.

We solve this differential equation in the same way we have solved the others in this paper: we define $\phi = C(1-u)^{-i\omega/3}F(u)$ and solve for $F(u)$ perturbatively in ω_5 and q_5. The result is

$$F(u) = 1 - \frac{i\omega_5}{3} f_1(u) + q_5^2 \left(-\frac{1}{4} f_1(u) - \frac{\sqrt{3}}{6} f_2(u) \right) . \tag{4.30}$$

The constant C can be re-expressed in terms of the boundary value of ϕ, $C = \phi_0 / F(0)$.

To calculate the two point function, we isolate the term in the action proportional to ϕ'^2:

$$S = -\frac{2^6 \pi^3 N^3 T^6}{3^7} \int du\, d^6x \frac{f}{u^2} \phi'^2 + \cdots . \tag{4.31}$$

From this expression, we find that the retarded Green's function is

$$G_{yz,yz} = -\frac{2^5 \pi^2 T^5 N^3}{3^6} \left(i\omega + \frac{3}{8\pi T} q^2 \right) . \tag{4.32}$$

According to [5], there is a Kubo formula which relates this Green's function to the shear viscosity η. The formula states that

$$\eta = -\lim_{\omega \to 0} \left[\lim_{q \to 0} \frac{1}{\omega} \mathrm{Im}\, G_{yz,yz} \right] = \frac{2^5 \pi^2}{3^6} N^3 T^5 \tag{4.33}$$

which matches (4.16).[2]

5. N-puzzles

The viscosities and diffusion constants calculated here present new N-puzzles, new questions about why particular N dependences arise in these M-brane theories. From the gravitational point of view, the N dependence comes from the gravitational coupling constant κ_{11}. What is missing is a direct understanding of how a field theoretic description of a stack of N M-branes gives rise to the same N dependences. While for D3-branes, the N dependence could be understood perturbatively from SU(N) gauge theory, for the M-branes, no equivalent of SU(N) gauge theory exists (as yet) as an alternate description. To review, the diffusion constants for the M-branes are all N independent (3.24), (3.36), (4.15), (4.26). The viscosities (4.16), (4.27) on the other hand have the same N dependence as the entropy (2.6) and (2.11) of the corresponding M2- or M5-brane theory. Note that the equation $D = \eta/(\epsilon + P)$ relates the N dependence of the viscosity to that of the

[2] There are still other ways of calculating the viscosity through AdS/CFT correspondence. For example, the authors of [13] related the graviton absorption cross section by non-extremal D3-branes to the viscosity of finite temperature $\mathcal{N} = 4$ SU(N) super Yang-Mills theory. Presumably a similar calculation could be done for the non-extremal M2- and M5-brane supergravity solutions.

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entropy and of the diffusion constant D. But there is still at least one new unexplained N dependence here.

	S	η	D	D_R
D3-branes	$(\pi^2/2)N^2T^3$	$(\pi/8)N^2T^3$	$(1/4\pi T)$	$(1/2\pi T)$
M5-branes	$(2^7\pi^3/3^6)N^3T^5$	$(2^5\pi^2/3^6)N^3T^5$	$(1/4\pi T)$	$(3/8\pi T)$
M2-branes	$(8\sqrt{2}\pi^2/27)N^{3/2}T^2$	$(2\sqrt{2}\pi/27)N^{3/2}T^2$	$(1/4\pi T)$	$(3/4\pi T)$

The first M-brane N-puzzle was the entropy puzzle, the N^3 dependence of the M5-brane entropy (and also the $N^{3/2}$ dependence of the M2-brane entropy) [11]. For the D3-branes, the N^2 dependence was more or less clearly the N^2 degrees of freedom of a SU(N) gauge theory [14], but what could provide N^3 degrees of freedom?

A less well known N-puzzle comes from the N dependence of the normalized three-point functions for M-branes. For D3-branes, the three-point functions scale as N^{-1}, as can be easily understood from 't Hooft counting. However, from [15], we know that the three-point functions for M5-branes scale as $N^{-3/2}$ while for M2-branes, as $N^{-3/4}$.

The original motivation for writing this paper arose precisely out of these N-puzzles, a feeling that these N dependences might provide a key to better understanding M-brane theories. In particular, the more N dependences we know, perhaps the better our chances of finding some pattern and unlocking the mysteries of M-theory.[3]

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